THE BORN-INFELD ELECTROMAGNETISM IN KALUZA-KLEIN THEORY

by

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Abstract We investigate the properties of non-linear electromagnetism based on the Born-Infeld Lagrangian in multi-dimensional theories of Kaluza-Klein type. We consider flat space-time solutions only, which means that the space-time metric is constant, and the only supplementary variable is the dilaton field, a scalar. We show that in the case of Kaluza-Klein theory, the Born-Infeld Lagrangian describes an interesting interaction between the electromagnetic and scalar fields, whose propagation properties are modified in a non-trivial manner.

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1. Introduction

It is well known since the article by G. Boillat ([1]) that the Born-Infeld theory of electromagnetism ([2, 3, 4]), although highly non-linear, leads under certain physical requirements to the propagation without birefringence, which makes this theory very special indeed. In addition, all types of disturbances and waves propagate without producing shocks in a finite time, implying that this case belongs to the class of theories called *completely exceptional*.

The Born-Infeld Lagrangian is defined as the square root of the following determinant (we insist on using the mixed covariant and contravariant indeces, because only then the corresponding expression can be identified with a *matrix*, i.e. a linear operator, to which the concept of the determinant does apply):

$$\mathcal{L} = \sqrt{\det(\delta_\mu{}^\nu + F_\mu{}^\nu)}\,, \tag{1}$$

where $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$ is the Maxwell tensor (partial derivatives in covariant indeces will be denoted by a comma). This Lagrangian can be written also as $\mathcal{L} = \sqrt{1 + 2P - S^2}$ where $P = \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ and $S = \frac{1}{4} {}^*F^{\mu\nu} F_{\mu\nu}$, with ${}^*F^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$ is the dual of $F_{\mu\nu}$. In terms of the electric and magnetic field intensities one has $P = \frac{1}{2}(\mathbf{B}^2 - \mathbf{E}^2)$ and $S = \mathbf{E} \cdot \mathbf{B}$. It has been proved in [1] that the only other possible completely exceptional theory comes from the singular Lagrangian $\mathcal{L} = P/S$.

It is amusing to note that the authors of this remarkable theory have been led by the considerations of finiteness of energy, a natural postulate of recovering Maxwell's theory as linear approximation, and the hope to find soliton-like solutions representing charged particles like in the non-linear (but non-relativistic) theory proposed by G. Mie ([5]), rather than by an appropriate character of wave propagation [7].

Another important property that the Born-Infeld theory shares with Maxwell's theory is its invariance under dual transformations, i.e., $SO(2)$ rotations of the fields $F_{\mu\nu}$ and ${}^*F^{\mu\nu}$ into each other (see e.g. [6]).

Originally, the Born-Infeld Lagrangian did not have any direct geometrical meaning. Another type of non-linear electrodynamics, with a clear geometrical rationale, which in the limit of weak field coincides with the Maxwell theory, but which differs from the Born-Infeld theory already in the second terms of its Taylor expansion, can be derived from the Kaluza-Klein theory in five dimensions, in the absence of scalar and gravitational fields. It is based on the fact that in five dimensions the Gauss-Bonnet term, $R_{ABCD}\, R^{ABCD} - 4\, R_{AB}\, R^{AB} + R^2$ is not a topological invariant like in the 4-dimensional case, but leads to non-trivial equations of motion of second order when added to the usual Einstein-Hilbert Lagrangian R. (The Gauss-Bonnet Lagrangian of the 5-dimensional Kaluza-Klein theory including the variable scalar field has been derived by F. Müller-Hoissen in [8].) The full Lagrangian is [9, 10]:

$$\mathcal{L} = R + \gamma\, (R_{ABCD}\, R^{ABCD} - 4\, R_{AB}\, R^{AB} + R^2)\,, \tag{2}$$

with γ being a dimensional parameter characterizing the relative weight of non-linear terms. When expressed in four dimensions in terms of the Maxwell tensor $F_{\mu\nu}$, one finds $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{3\gamma}{16} (F_{\mu\nu} F^{\mu\nu})^2 - 2 (F_{\mu\lambda} F_{\nu\rho} F^{\mu\nu} F^{\lambda\rho})$, $(\mu, \nu = 0, 1, 2, 3)$. In terms of the invariants P and S this Lagrangian is given by $\mathcal{L} = 2P + \frac{3\gamma}{2} S^2$, which with an appropriate choice of γ yields (up to a constant) the square of the Born-Infeld Lagrangian. This theory will be explored in section 5.

Recently, the Born-Infeld type Lagrangians appeared naturally in string theories; this circumstance explains the revival of interest in the non-linear Born-Infeld electromagnetism and theories akin to it. For example, the interaction between the electromagnetic and scalar fields should be investigated because the dilaton (scalar) fields appear naturally in all effective string Lagrangians. Deser *et al.* [6] have considered a theory based on the Lagrangian $\mathcal{L} = \sqrt{\det(\delta_\mu{}^\nu + F_\mu{}^\nu + \phi_{,\mu} \phi^{,\nu})}$ and have found that such theories do not ensure the absence of shock propagation. For the theory to be completely exceptional the total Lagrangian must be a sum of the Born-Infeld Lagrangian (or Maxwell's theory, or the singular P/S theory mentioned above) and a Lagrangian depending on the invariant $\phi_{,\mu} \phi^{,\mu}$) only, meaning the absence of any interaction between those fields.

In spite of its very special properties, the Born-Infeld Lagrangian cannot be considered as the ultimate theory; the massless vector field it describes is certainly interacting with other possible excitations of string theories, in particular with the scalar field. We believe that one of the best ways of introducing the invariant interactions with the scalar field is the dimensional reduction technique.

Instead of adding certain terms (out of several possible ones) containing the scalar field alone or interacting with the electromagnetic field, like in [6], we shall compute the Born-Infeld Lagrangian directly in $D + 1 + 1$ dimensions (with $D = 1, 2, 3$ being the number of space-like dimensions), which after reduction to $D + 1$ space-time dimensions will give us a unique expression containing the terms describing the contributions of the fields and their interactions. We then study the propagation of dynamical fields in this setting. Although the full theory is very highly non-linear and leads to very complicated characteristic cone equations, it is possible to get some results in the case when one of the fields is constant (i.e. does not propagate), but still can influence the propagation of the other one; partial results can be also obtained in the weak field limit.

Thus, we shall investigate systematically the combination of the Born-Infeld and Kaluza-Klein theories in lower dimensions, starting from the total dimension 3, i.e. with one space and one time coordinate, plus one extra Kaluza-Klein dimension. We also analyze total dimensions 4 and 5. In general, the determinant of an arbitrary $N \times N$ matrix is a polynomial of N-th order containing all possible products of the elements of the matrix. However, we shall see that due to the particular structure of the matrix $\delta_\mu{}^\nu + F_\mu{}^\nu$, its determinants computed in odd dimensions $N = 2k + 1$ yield polynomials of even order $2k$ only.

2. Characteristic surfaces and field propagation

Our aim is to find out how do propagate the fields $F_{\mu\nu}$ and ϕ, i.e. how the characteristic surfaces (cones) depend on dimension, whether both $F_{\mu\nu}$ and ϕ propagate along the same cones, and whether birefringence occurs or not, i.e., are these cones unique. The systems under consideration are second order partial differential equations, linear in the highest derivatives, with coefficients which depend only on the fields themselves. It is known (see e.g. [11]) that the systems of this type can be reduced to a set of equations of first order in the derivatives via introduction of auxiliary fields. These auxiliary variables are the independent linear combinations of the first partial derivatives of functions describing the degrees of freedom of our system. For a scalar field these will be just all its first derivatives, $\partial_0\phi \equiv \psi$, $\partial_i\phi \equiv \chi_i$ (i representing the spatial indeces); for the electromagnetic potential A_μ the auxiliary variables are the electric field E_i and magnetic field B_i which are the only independent combinations of first order derivatives of A_μ.

Next we represent the differential system by means of a matrix whose entries contain the operators of partial derivation or multiplicative coefficients, acting on a vector-column representing auxiliary fields. If the vector-column $\mathbf{u}$ with the fields ψ, χ_i, E_i and B_i contains N elements, then let us denote by $\mathcal{A}$ the NxN matrix containing the partial derivatives and by $\mathcal{B}$ the NxN matrix containing the multiplicative factors. Then the field equations can be written in the form

$$\mathcal{A}^\mu(\mathbf{u})\partial_\mu\mathbf{u} + \mathcal{B}(\mathbf{u})\mathbf{u} = 0\,. \tag{3}$$

If the hypersurface defined by the implicit equation

$$\Sigma\left(x^\mu\right) = 0\,. \tag{4}$$

is a surface of discontinuity, then the first derivatives of fields are discontinuous across this surface, whereas the fields themselves are continuous. So, when applied to the *discontinuities* across the hypersurface (4), the equation (3) reduces to

$$\left(\mathcal{A}^\mu\,\Sigma_\mu\right)\delta_1\mathbf{u} = 0\,, \tag{5}$$

where $\Sigma_\mu \equiv \partial_\mu\Sigma$, and $\delta_1\mathbf{u}$ denotes the discontinuity of the first derivative across Σ, $\delta_1\mathbf{u} \equiv \partial\mathbf{u}/\partial\Sigma|_+ - \partial\mathbf{u}/\partial\Sigma|_-$. By definition, for a characteristic surface one has $\delta_1\mathbf{u} \neq 0$, therefore, in order for (5) to hold, one must have

$$\det\Big(\mathcal{A}^\mu\,\Sigma_\mu\Big) = 0\,, \tag{6}$$

on the surface of discontinuity. The characteristic equation (6) determines the surface whose generic equation is $H(x, \Sigma_\mu) = 0$, with H a homogeneous function of order N in Σ_μ. The theories we study are *completely exceptional* since they obey the corresponding condition of [1], namely $\delta_0 H \equiv H|_+ - H|_- = 0$ (see also [6]).

We will use this formalism to study the propagation of waves, characteristic equations and the possibility of birefringence in various theories. The solutions of the characteristic equation define the hypersurfaces along which the propagation takes place. If the solution is unique, it is said that there is no birefringence.

To make this setup clearer, let us show on a simple example how these ideas work. We start with the simplest possible case: scalar field wave equation in a two-dimensional space-time (t, x):

$$\partial_0^2\, \phi - \partial_x^2\, \phi = 0\,. \tag{7}$$

(Partial derivatives in Lorentz indeces, $(0, x, y, z)$ or $(0, 1, 2, 3)$, will be denoted by ∂). According to the prescription, we can use as auxiliary fields ψ and χ the first derivatives of the scalar field ϕ, $\partial_0\phi = \psi$ and $\partial_x\phi = \chi$. Then by definition, the first derivatives of auxiliary fields are not independent, because we have, as $\partial_0\,(\partial_x\,\phi) = \partial_x\,(\partial_0\,\phi)$, automatically $\partial_0\chi - \partial_x\psi = 0$. On the other hand the dynamical equation (7) can be written as $\partial_0\psi - \partial_x\chi = 0$. In the matrix notation of (3) these two equations can be combined to yield

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_0 \begin{pmatrix} \psi \\ \chi \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \partial_x \begin{pmatrix} \psi \\ \chi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{8}$$

We then find

$$\mathcal{A}^\mu \Sigma_\mu = \begin{pmatrix} -\Sigma_x & \Sigma_0 \\ \Sigma_0 & -\Sigma_x \end{pmatrix}, \tag{9}$$

and the characteristic equation $\det(\mathcal{A}^\mu\Sigma_\mu) = 0$ can be written as

$${\Sigma_0}^2 - {\Sigma_x}^2 = 0\,, \tag{10}$$

where $\Sigma_0 \equiv \partial_0\Sigma$ and $\Sigma_x \equiv \partial_x\Sigma$. This last equation defines the characteristic surfaces $\Sigma(t, x)$, which in this case are the light-cones in two space-time dimensions.

The same technique can be easily applied to the electromagnetic Maxwellian field. The matrix becomes quite cumbersome, because we have now six independent combinations of its first derivatives (the fields $\mathbf{E}$ and $\mathbf{B}$) appearing in the first-order equations of the Maxwell system. As we know, the characteristic surfaces in four dimensions are given by $\Sigma_{,\mu}\,\Sigma^{,\mu} = {\Sigma_0}^2 - {\Sigma_x}^2 - {\Sigma_y}^2 - {\Sigma_z}^2 = 0$. In the next sections we apply these techniques to more complex cases in various dimensions.

3. Born-Infeld type Lagrangians in two dimensions

3.1 The Born-Infeld theory

Starting from the lowest-dimensional case where there is only one space and one time coordinate one finds that $\det({\delta_\mu}^\nu + {F_\mu}^\nu)$ has the form

$$\det \begin{pmatrix} 1 & {F^0}_x \\ {F^x}_0 & 1 \end{pmatrix} = 1 - {F^0}_x\,{F^x}_0. \tag{11}$$

Since $F_{\mu\nu} = -F_{\nu\mu}$, the Born-Infeld Lagrangian can be written as

$$\mathcal{L} = \sqrt{1 + 2P}, \tag{12}$$

where

$$P \equiv \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{13}$$

is the only invariant in two dimensions. The Euler-Lagrange equations are then

$$\mathcal{L}^{\alpha\beta}{}_{,\alpha} = 0, \tag{14}$$

with $\mathcal{L}^{\alpha\beta} = \mathcal{L}_P F_{\alpha\beta}$ and $\mathcal{L} = (\partial\mathcal{L}/\partial P)$. The equation (14) can be written as

$$(1 + 2P)\, F^{\alpha\beta}{}_{,\beta} - (F_{\mu\nu}F^{\mu\nu})_{,\beta}\, F^{\alpha\beta} = 0\,. \tag{15}$$

In the linear approximation, when the second term of the above equation can be neglected, we get Maxwell's equations in two dimensions, i.e. $F^{\alpha\beta}{}_{,\beta} = 0$. The electric field is defined by $F^{0x} = \partial_x A_0 - \partial_0 A_x = E$, and the magnetic field does not exist. The equations for E are:

$$\partial_0 E = 0, \qquad \partial_x E = 0. \tag{16}$$

We see that the only solution is $E =$ const. and there are no waves in Born-Infeld (nor in Maxwell) theory in a two-dimensional space-time.

3.2 Born-Infeld theory in Kaluza-Klein space

With a compactified extra Kaluza-Klein dimension over the two-dimensional spacetime, and with constant metric tensor, we get the following expression for $\det(\delta_\mu{}^\nu + F_\mu{}^\nu)$ containing the contribution of the scalar field:

$$\begin{aligned}\det\begin{pmatrix} 1 & F^0{}_x & \partial^0\phi \\ F^x{}_0 & 1 & \partial^x\phi \\ \partial_0\phi & \partial_x\phi & 1 \end{pmatrix} &= 1 - F^0{}_x\, F^x{}_0 - \partial_0\phi\partial^0\phi - \partial_x\phi\partial^x\phi = \\ &= 1 + \tfrac{1}{2}F_{\mu\nu}F^{\mu\nu} - \phi_{,\mu}\phi^{,\mu}\,. \end{aligned} \tag{17}$$

In the above expression one can note the absence of direct coupling between the scalar and gauge fields. Taking the square root of (17) one finds the Born-Infeld Lagrangian :

$$\mathcal{L} = \sqrt{1 + 2P - 2\Phi} \tag{18}$$

where P is given by (13) and

$$\Phi \equiv \frac{1}{2}\phi_{,\mu}\phi^{,\mu}\,. \tag{19}$$

The equations of motion for the vector potential A_μ and the scalar ϕ are, respectively

$$\mathcal{L}^2\, F^{\alpha\beta}{}_{,\beta} - \frac{1}{4}F^{\alpha\beta}\,(F_{\mu\nu}F^{\mu\nu})_{,\beta} + \frac{1}{2}F^{\alpha\beta}\,(\phi^{,\mu}\phi_{,\mu})_{,\beta} = 0, \tag{20}$$

and

$$\mathcal{L}^2\phi^{,\alpha}{}_{,\alpha} - \frac{1}{4}\phi^{,\alpha}(F_{\mu\nu}F^{\mu\nu})_{,\alpha} + \frac{1}{2}\phi^{,\alpha}(\phi^{,\mu}\phi_{,\mu})_{,\alpha} = 0. \tag{21}$$

As before, we define the electric field E as the derivative of the potential A_μ

$$\partial_x A_0 - \partial_0 A_x = E\,, \tag{22}$$

and the two auxiliary fields ψ and χ which are the first partial derivatives of the field ϕ,

$$\partial_0\phi = \psi, \qquad \partial_x\phi = \chi\,. \tag{23}$$

The equation (23) implies

$$\partial_0\chi - \partial_x\psi = 0\,. \tag{24}$$

Using (20) and (22) we obtain the equation of motion for the electric field

$$(1-\psi^2+\chi^2)\,\partial_0 E + \psi\,E\,\partial_0\psi - \chi\,E\,\partial_0\chi = 0\,, \tag{25}$$

and an analogous equation with ∂_0 replaced by ∂_x. From (21) and (23) we get the equation of motion for the scalar field :

$$(1+E^2+\chi^2)\,\partial_0\psi - (1+E^2-\psi^2)\,\partial_x\chi - \psi\,\chi\,(\partial_x\psi+\partial_0\chi) + \psi\,E\,\partial_0 E - \chi\,E\,\partial_x E = 0\,. \tag{26}$$

Using the formalism introduced in section 2 we define the column-vector $\mathbf{u}$ with the fields E, ψ and χ. Then the equations (24), (25) and (26) can be put in the following matrix form:

$$\begin{pmatrix} 1-\psi^2+\chi^2 & \psi\,E & -\chi\,E \\ 0 & 0 & 1 \\ \psi\,E & 1-E^2+\chi^2 & -\psi\,\chi \end{pmatrix} \partial_0 \begin{pmatrix} E \\ \psi \\ \chi \end{pmatrix} + \\ + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ -\chi\,E & -\psi\,\chi & -(1-E^2-\psi^2) \end{pmatrix} \partial_x \begin{pmatrix} E \\ \psi \\ \chi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{27}$$

¿From the equations (27), (3) and (5) we find

$$\mathcal{A}^\mu\,\Sigma_\mu = \\ \begin{pmatrix} (1-\psi^2+\chi^2)\Sigma_0 & \psi E\Sigma_0 & -\chi E\Sigma_0 \\ 0 & -\Sigma_x & \Sigma_0 \\ \psi E\Sigma_0 - \chi E\Sigma_x & (1-E^2+\chi^2)\Sigma_0 - \psi\chi\Sigma_x & -\psi\chi\Sigma_0 - (1-E^2-\psi^2)\Sigma_x \end{pmatrix} \tag{28}$$

Therefore the characteristic equation $\det(\mathcal{A}^\mu\Sigma_\mu) = 0$ is :

$$\begin{aligned} \Sigma_0 \Big\{ &[(1-\psi^2+\chi^2)(1-E^2+\chi^2) - \psi^2\,E^2]\,{\Sigma_0}^2 \\ &- [(1-\psi^2+\chi^2)(1-E^2-\psi^2) + \chi^2\,E^2]\,{\Sigma_x}^2 \\ &-2\psi\chi\;[(1-\psi^2+\chi^2) - E^2]\,\Sigma_0\Sigma_x \Big\} = 0\,. \end{aligned} \tag{29}$$

The equation (29) can be written in the following covariant form:

$$\begin{aligned} &(V^\gamma\Sigma_\gamma)\Big[(1-\phi_{,\beta}\phi^{,\beta})(g^{\mu\nu} + F^{\mu\alpha}{F^\nu}_\alpha) \\ &-(1-\phi_{,\beta}\phi^{,\beta})(\phi_{,\alpha}\phi^{,\alpha}g^{\mu\nu} - \phi^{,\mu}\phi^{,\nu}) + \tfrac{1}{2}F_{\alpha\beta}F^{\alpha\beta}\phi^{,\mu}\phi^{,\nu}\Big]\,\Sigma_\mu\Sigma_\nu = 0, \end{aligned} \tag{30}$$

with V^γ a time-like vector. We see that the characteristic equation (30) separates into a part that does not propagate the electric field and a part that propagates the scalar field. Consequently, in a linear theory given by the Lagrangian (17) (without the square root), the electric field E does not propagate and the scalar field propagates along the Minkowskian light-cones, $g^{\mu\nu}\Sigma_\mu\Sigma_\nu = 0$ or in Lorentzian coordinates $\Sigma_0^2 - \Sigma_x^2 = 0$, independently of the constant value of the field E.

To find the dispersion relation of the theory we put $\Sigma_\mu \equiv \Sigma_{,\mu}$, and use the ansatz $\Sigma = a e^{i(\omega t - kx)}$. Inserting into (29) we find the relation $k = k(\omega)$ characterizing the propagation of the scalar field in this theory :

$$k = \pm\omega \frac{\sqrt{\beta^2 + \alpha\gamma} \pm \beta}{\alpha}, \tag{31}$$

where $\alpha \equiv (1 - \psi^2 + \chi^2)(1 - E^2 - \psi^2) + E^2\chi^2$, $\beta = \psi\chi\,[(1 - \psi^2 + \chi^2) - E^2]$, and $\gamma = (1 - \psi^2 + \chi^2)(1 - E^2 + \chi^2) - E^2\psi^2$. To simplify this expression let us consider a scalar wave propagating in the vacuum, $E = 0$. Then (31) reduces to

$$k = \pm\omega \frac{\sqrt{1 - \psi^2 + \chi^2} \pm \psi\chi}{1 - \psi^2}. \tag{32}$$

In the weak field limit $\psi^2 \sim \chi^2 \ll 1$ one has $k = \pm\omega \left[1 + \frac{1}{2}(\psi \pm \chi)^2\right]$. This means that the propagation velocity of the wave, $v \equiv \mid \omega/k \mid = 1/\left[1 + \frac{1}{2}(\psi \pm \chi)^2\right]$, is less than one, i.e., less than the speed of light in the linear theory. We also note that (32) gives no birefringence, as it should be expected for a a scalar field. There are only two solutions, one describing an outgoing, the other an ingoing wave.

4. Born-Infeld type Lagrangians in 3 dimensions

4.1 Maxwell's theory

In three space-time dimensions Maxwell's theory is non-trivial. The Lagrangian is $\mathcal{L} = 2P$, where $P = \frac{1}{4} F_{\mu\nu}F^{\mu\nu}$. Introducing the four-potential A_μ we define the Maxwell tensor as before,

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}. \tag{33}$$

In order to simplify the problem we consider here the characteristic equation for a system depending only on two variables t and y, and in which the only non-vanishing fields are $A_x(t, y)$, $E_x(t, y)$ and $B(t, y)$. Despite of this simplified choices for the potentials (some of the components suppressed, and depending only on some variables) the result is covariant and unique. For this type of simplifications see e.g. [11, 12]. From the equation (33), and defining $F^{0x} = E_x$, $F^{xy} = B$, we have $\partial_0 A_x + E_x = 0$ and $\partial_y A_x + B = 0$. The two equations combined yield

$$\partial_0 B - \partial_y E_x = 0\,. \tag{34}$$

The dynamical equation of motion is

$$\partial_y B - \partial_0 E_x = 0. \tag{35}$$

Then, using the formalism of section 2 we obtain

$$\mathcal{A}^{\mu}\Sigma_{\mu} = \begin{pmatrix} -\Sigma_y & \Sigma_0 \\ -\Sigma_0 & \Sigma_y \end{pmatrix}\,, \tag{36}$$

so the characteristic equation $\det\left(\mathcal{A}^{\mu}\Sigma_{\mu}\right) = 0$ becomes

$$\Sigma_0^2 - \Sigma_y^2 = 0\,.$$

Its invariant generalization taking into account the two space-like dimensions is obvious :

$$\Sigma_0^2 - \Sigma_x^2 - \Sigma_y^2 = 0\,. \tag{37}$$

Therefore, also in three space-time dimensions, Maxwell's theory leads to the propagation along the light-cones.

4.2 Maxwell's theory with non-interacting scalar field

Before we proceed to study Born-Infeld itself in 3D, and Born-Infeld with an extra Kaluza-Klein dimension let us study a non-interacting Maxwell and scalar field theory whose Lagrangian is given by

$$\mathcal{L} = 2P - 2\Phi\,, \tag{38}$$

where, as before, $P = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ and $\Phi = \frac{1}{2}\phi_{,\alpha}\,\phi^{,\alpha}$. Variation with respect to the electromagnetic potential A_ρ yields Maxwell's equations $F^{\rho\sigma}{}_{,\rho} = 0$ whereas variation with respect to ϕ yields $\phi^{,\rho}{}_{,\rho} = 0$. In order to find the characteristic equations we again suppose that the system depends only on two variables (t, y), with nonzero fields given by $A_x(t,y)$, $E_x(t,y)$, $B(t,y)$ and $\phi(t,y)$. Then with the definitons $\partial_0\phi - \psi = 0$ and $\partial_y\phi - \chi = 0$, we have the following set of equations,

$$\begin{aligned} -\partial_0 B + \partial_y E_x = 0\,, \qquad & -\partial_0 E_x + \partial_y B = 0\,, \\ \partial_0\chi - \partial_y\psi = 0\,, \qquad & \partial_0\psi - \partial_y\chi = 0\,. \end{aligned} \tag{39}$$

Calculating $\mathcal{A}^{\mu}\Sigma_{\mu}$ one obtains

$$\mathcal{A}^{\mu}\Sigma_{\mu} = \begin{pmatrix} \Sigma_y & -\Sigma_0 & 0 & 0 \\ -\Sigma_0 & \Sigma_y & 0 & 0 \\ 0 & 0 & -\Sigma_y & \Sigma_0 \\ 0 & 0 & \Sigma_0 & -\Sigma_y \end{pmatrix}\,. \tag{40}$$

The characteristic equation $\det(\mathcal{A}^{\mu}\Sigma_{\mu}) = 0$ is $(\Sigma_0^2 - \Sigma_y^2)(\Sigma_0^2 - \Sigma_y^2) = 0\,.$

Again, the three-dimensional generalization is obvious:

$$(\Sigma_0^2 - \Sigma_x^2 - \Sigma_y^2)(\Sigma_0^2 - \Sigma_x^2 - \Sigma_y^2) = 0\,. \tag{41}$$

We see that the characteristic is the equation for two non-interacting fields propagating with the velocity of light. The propagation of the two fields splits into two independent parts because the characteristic matrix (40) is block-diagonal.

4.3 The Born-Infeld theory

In a three-dimensional space-time the Born-Infeld determinant is :

$$\det\begin{pmatrix} 1 & F^0{}_x & F^0{}_y \\ F^x{}_0 & 1 & F^x{}_y \\ F^y{}_0 & F^y{}_x & 1 \end{pmatrix} = 1 - F^0{}_x F^x{}_0 - F^0{}_y F^y{}_0 - F^x{}_y F^y{}_x = \\ = 1 + F_{\mu\nu}F^{\mu\nu}\,, \tag{42}$$

where we note the absence of terms of third order in the fields. Therefore, taking the square root, we get the same action as in two dimensions, i.e. $\mathcal{L} = \sqrt{1+2P}$ but now $F^{0x} = E_x$, $F^{0y} = E_y$ and $F^{xy} = B$ as in Maxwell's theory. The dynamical equation of motion, given by $\mathcal{L}^{\alpha\beta}{}_{,\alpha} = 0$ with $\mathcal{L}^{\alpha\beta} = \mathcal{L}_P F^{\alpha\beta}$ yields

$$(1+2P)F^{\alpha\beta}{}_{,\alpha} - \frac{1}{4}(F_{\mu\nu}F^{\mu\nu})_{,\alpha}F^{\alpha\beta} = 0\,. \tag{43}$$

Combining (33) and (43), we get the following set of six equations,

$$\begin{gathered} \partial_x A_0 - \partial_0 A_x - E_x = 0\,, \\ \partial_y A_0 - \partial_0 A_y - E_y = 0\,, \\ \partial_x A_y - \partial_y A_x - B = 0\,, \\ (1+B^2-E_y^2)\partial_x E_x + (1+B^2-E_x^2)\partial_y E_y + \\ E_xE_y\partial_x E_y - E_xB\partial_x B + E_xE_y\partial_y E_x - E_yB\partial_y B = 0\,, \\ (1-E_x^2-E_y^2)\partial_y B - (1+B^2-E_y^2)\partial_0 E_x - \\ E_xE_y\partial_0 E_y + E_xB\partial_0 B + E_xB\partial_y E_x + E_yB\partial_y E_y = 0\,, \\ (1-E_x^2-E_y^2)\partial_x B + (1+B^2-E_x^2)\partial_0 E_y + \\ E_xE_y\partial_0 E_x - E_yB\partial_0 B + E_xB\partial_x E_x + E_yB\partial_x E_y = 0\,. \end{gathered} \tag{44}$$

As we did in the previous section, where we have chosen a reduced system, we shall study the characteristics for a system depending only on two variables (t,y) and with nonzero fields given by $A_x(t,y)$, $E_x(t,y)$ and $B(t,y)$. In this case in (44) only two independent equations remain :

$$\begin{gathered} \partial_0 B - \partial_y E_x = 0\,, \\ (1-E_x^2)\partial_y B - (1+B^2)\partial_0 E_x + E_xB\partial_0 B + E_xB\partial_y E_x = 0\,. \end{gathered} \tag{45}$$

Using the formalism of section 2 we obtain

$$\mathcal{A}^\mu\Sigma_\mu = \begin{pmatrix} -\Sigma_y & \Sigma_0 \\ -(1+B^2)\Sigma_0 + E_xB\Sigma_y & E_xB\Sigma_0 + (1+E_x^2)\Sigma_y \end{pmatrix}\,. \tag{46}$$

Thus the characteristic equation $\det(\mathcal{A}^\mu\Sigma_\mu) = 0$ can be put in the form,

$$\Sigma_0^2 - \Sigma_y^2 + (B\Sigma_0 - E_x\Sigma_y)^2 = 0\,. \tag{47}$$

In 3D the dual to $F_{\alpha\beta}$ is

$$ {}^*F^\gamma = \frac{1}{2}\,\epsilon^{\alpha\beta\gamma} F_{\alpha\beta}, \tag{48} $$

where $\epsilon^{\alpha\beta\gamma} = \frac{1}{\sqrt{-g}} e^{\alpha\beta\gamma}$ is the Levi-Cività tensor and $e^{\alpha\beta\gamma}$ the Levi-Cività symbol.

One can write the characteristic equation (47) in a covariant form,

$$ (g^{\alpha\beta} + {}^*F^\alpha\, {}^*F^\beta)\Sigma_\alpha \Sigma_\beta = 0\,. \tag{49} $$

Since $\Sigma_\mu \equiv \Sigma_{,\mu}$, for some function Σ, one can seek the ansatz $\Sigma = a e^{i(\omega t - ky)}$ to find with the help of (47) the covariant expression

$$ k = \pm\omega \frac{\sqrt{1 + B^2 - E^2} \mp EB}{1 - E^2}\,, \tag{50} $$

with $E = \sqrt{E_x^2 + E_y^2}$. This means that the propagation velocity is less than the speed of light in the linear theory, and there is no birefringence as it is expected for a Born-Infeld theory, [1].

4.4 Born-Infeld theory in Kaluza-Klein space

Adding the extra Kaluza-Klein dimension gives:

$$ \det \begin{pmatrix} 1 & F^0{}_x & F^0{}_y & \partial^0\phi \\ F^x{}_0 & 1 & F^x{}_y & \partial^x\phi \\ F^y{}_0 & F^y{}_x & 1 & \partial^y\phi \\ \partial_0\phi & \partial_x\phi & \partial_y\phi & 1 \end{pmatrix} = $$

$$ 1 + \tfrac{1}{2} F_{\mu\nu}F^{\mu\nu} - \phi_{,\mu}\phi^{,\mu} - \phi_{,\mu}\phi_{,\nu}(\tfrac{1}{2}F_{\alpha\beta}F^{\alpha\beta}g^{\mu\nu} - F^{\mu\alpha}F^\nu{}_\alpha)\,. \tag{51} $$

The Born-Infeld type Lagrangian is then

$$ \mathcal{L} = \sqrt{1 + 2P - 2\Phi - 2I}\,, \tag{52} $$

where $P = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, $\Phi = \frac{1}{2}\phi_{,\mu}\phi^{,\mu}$ and $I = \frac{1}{2}\phi_{,\mu}\phi_{,\nu}(\frac{1}{2}F_{\alpha\beta}F^{\alpha\beta}g^{\mu\nu} - F^{\mu\alpha}F^\nu{}_\alpha)$. The variation with respect to the vector potential A_σ yields

$$ \begin{aligned} &\mathcal{L}^2\left[1 - (\phi_{,\mu}\phi^{,\mu})\right]F^{\alpha\beta}{}_{,\alpha} - \left[\mathcal{L}\mathcal{L}_{,\alpha}(1 - \phi_{,\mu}\phi^{,\mu}) + \mathcal{L}^2(\phi_{,\mu}\phi^{,\mu})_{,\alpha}F^{\alpha\beta}\right] + \\ &2\mathcal{L}^2\phi^{,\alpha}\phi_{,\lambda}F^{\lambda\beta}{}_{,\alpha} + \\ &2\left[\mathcal{L}^2\phi^{,\alpha}\phi_{,\lambda,\alpha} + \mathcal{L}^2\phi^{,\alpha}{}_{,\alpha}\phi_{,\lambda} - \mathcal{L}\mathcal{L}_{,\alpha}\phi^{,\alpha}\phi_{,\lambda}\right]F^{\lambda\beta} = 0\,. \end{aligned} \tag{53} $$

and the variation with respect to the scalar field ϕ gives

$$ \begin{aligned} &\mathcal{L}^2(1 + \tfrac{1}{2}F_{\mu\nu}F^{\mu\nu})\phi^{,\alpha}{}_{,\alpha} - \mathcal{L}\mathcal{L}_{,\alpha}\phi^{,\alpha} + \mathcal{L}^2\phi^{,\alpha}(\tfrac{1}{2}F_{\mu\nu}F^{\mu\nu})_{,\alpha} \\ &-\mathcal{L}^2\left[(F^{\alpha\sigma}F^\lambda{}_\sigma)_{,\alpha}\phi_{,\lambda} + F^{\alpha\sigma}F^\lambda{}_\sigma\phi_{,\lambda,\alpha}\right] \\ &-\mathcal{L}\mathcal{L}_{,\alpha}(\tfrac{1}{2}F_{\mu\nu}F^{\mu\nu}\phi^{,\alpha} - F^{\alpha\sigma}F^\lambda{}_\sigma\phi_{,\lambda}) = 0\,. \end{aligned} \tag{54} $$

Again, we choose a reduced system to study the characteristic equations for the case depending only on two variables (t, y) and with nonzero fields given by $A_x(t, y)$,

$E_x(t,y)$ $B(t,y)$, and $\phi(t,y)$, with $E_x = -\partial_0 A_x$, $B = -\partial_y A_x$, $\psi = \partial_0\phi$ and $\chi = \partial_y\phi$. The invariants are now $P = \frac{1}{2}(B^2 - E^2)$, $\Phi = \frac{1}{2}(\psi^2 - \chi^2)$ and $I = \frac{1}{2}(E\chi - B\psi)^2$.

Now we can develop the formalism of section 2, but the high degree of non-linearity makes the problem almost intractable. The defining equations are still the same, very simple. The complete set of four equations is

$$
\begin{gathered}
-\partial_0 B + \partial_y E = 0\,,\\
\mathcal{L}^2\left[1-(\psi^2-\chi^2)\right](\partial_0 E_x - \partial_y B)\\
-E_x\left[\mathcal{L}\partial_0\mathcal{L}(1-(\psi^2-\chi^2)) + \mathcal{L}^2\partial_0(\psi^2-\chi^2)\right]\\
+B\left[\mathcal{L}\partial_y\mathcal{L}(1-(\psi^2-\chi^2)) + \mathcal{L}^2\partial_y(\psi^2-\chi^2)\right]\\
+2\mathcal{L}^2\left(\partial_0\psi^2 E_x - \psi\chi\partial_0 B - \psi\chi\partial_y E_x + \chi^2\partial_y B\right)\\
+2E_x\left[\mathcal{L}^2(2\psi\partial_0\psi - \chi\partial_y\psi - \psi\partial_y\chi) - \mathcal{L}\partial_0\mathcal{L}\psi^2 + \mathcal{L}\partial_y\mathcal{L}\psi\chi\right]\\
-2B\left[2\mathcal{L}^2(-2\chi\partial_0\chi + \psi\partial_0\chi + \chi\partial_0\psi) - \mathcal{L}\partial_0\mathcal{L}\psi\chi + \mathcal{L}\partial_y\mathcal{L}\chi^2\right] = 0\,,\\
\partial_0\chi - \partial_y\psi = 0\,,\\
-\mathcal{L}^2\left(1+(B^2-E^2)\right)(\partial_0\psi - \partial_y\chi) +\\
\psi\mathcal{L}\partial_0\mathcal{L} - \chi\mathcal{L}\partial_y\mathcal{L} - \mathcal{L}^2\psi\partial_0(B^2-E^2) + \mathcal{L}^2\partial_y\chi(B^2-E^2)\\
+\mathcal{L}^2\Big[-\partial_0({E_x}^2)\psi + \partial_0(E_x B)\chi - \partial_y(E_x B)\psi\\
-\partial_y(B^2)\chi - {E_x}^2\partial_0\psi + E_x B\partial_0\chi + E_x B\partial_y\psi - B^2\partial_y\chi\Big]\\
+\mathcal{L}\partial_0\mathcal{L}\Big[\psi(B^2-E^2) - (-{E_x}^2\psi + E_x B\chi)\Big]\\
+\mathcal{L}\partial_y\mathcal{L}\left[-\chi(B^2-E^2) - (E_x B\psi - B^2\chi)\right] = 0\,.
\end{gathered}
\tag{55}
$$

Then $\mathcal{A}^\mu\Sigma_\mu$ has the following structure,

$$
\mathcal{A}^\mu\Sigma_\mu = \begin{pmatrix}
\Sigma_y & -\Sigma_0 & 0 & 0\\
a_1\Sigma_0 - a_5\Sigma_y & a_2\Sigma_0 - a_6\Sigma_y & a_3\Sigma_0 - a_7\Sigma_y & -a_4\Sigma_0 + a_8\Sigma_y\\
0 & 0 & -\Sigma_y & \Sigma_0\\
b_1\Sigma_0 - b_5\Sigma_y & -b_2\Sigma_0 + b_6\Sigma_y & b_3\Sigma_0 - b_7\Sigma_y & -b_4\Sigma_0 - b_8\Sigma_y
\end{pmatrix}\,, \tag{56}
$$

where the coefficients a's and b's are known functions of E, B, ψ and χ. We can see how, due to the non-linearity, the off-diagonal terms start to appear signaling the interaction between the electromagnetic and scalar fields. Now $\det(\mathcal{A}^\mu\Sigma_\mu) = 0$ yields the following characteristic equation which we put in covariant notation

$$
G^{\alpha\beta\gamma\delta}\Sigma_\alpha\Sigma_\beta\Sigma_\gamma\Sigma_\delta = 0\,, \tag{57}
$$

with $G^{\alpha\beta\gamma\delta} = {g_1}^{\alpha\beta}{f_1}^{\gamma\delta} + {g_2}^{\alpha\beta}{f_2}^{\gamma\delta}$, where the tensor ${g_1}^{\alpha\beta}$ is given by

$$
\begin{gathered}
{g_1}^{\alpha\beta} = (\eta^{\alpha\beta} + {}^*F^{\alpha *}F^\beta - \phi_{,\mu}\phi^{,\mu}\eta^{\alpha\beta}) +\\
\Big[(2\phi^{,\alpha}\phi^{,\beta} - \phi_{,\mu}\phi^{,\mu}\eta^{\alpha\beta})(1+2Q-2I) - 2P\phi^{,\alpha}\phi^{,\beta} + +2I\eta^{\alpha\beta} + {}^*F^\alpha\phi^{,\beta}\sqrt{2I}\Big]\\
+\sqrt{2I}\sqrt{2\Phi}{}^*F^\alpha\phi^{,\beta}\,,
\end{gathered}
\tag{58}
$$

the tensor ${g_2}^{\alpha\beta}$ is given by

$$
\begin{aligned}
{g_2}^{\alpha\beta} = -\{ & F^\alpha \phi^{,\beta} + \left[-\phi_{,\mu}\phi^{,\mu} F^\alpha \phi^{,\beta} + 2F^\mu \phi_{,\mu} (\phi^{,\alpha}\phi^{,\beta} - \phi_{,\mu}\phi^{,\mu}\eta^{\alpha\beta}) \right] \\
& + \left[2F^\mu \phi_{,\mu}(1+2P) - \sqrt{2I}\tfrac{1}{2}{}^*F^\mu F_\mu \right] \eta^{\alpha\beta} \\
& - \left[2(2I) F^\alpha \phi^{,\beta} + \sqrt{2I}{}^*F^\mu \phi_{,\mu}(\phi_{,\nu}\phi^{,\nu})\eta^{\alpha\beta} \right] \} \,,
\end{aligned} \tag{59}
$$

${f_1}^{\gamma\delta}$ is

$$
\begin{aligned}
{f_1}^{\gamma\delta} = & (1+2P-2I)\eta^{\gamma\delta} - (\phi_{,\mu}\phi^{,\mu}\eta^{\gamma\delta} - \phi^{,\gamma}\phi^{,\delta}) \\
& + (1+2P-2\Phi){}^*F^{\gamma *}F^\delta - 2\sqrt{2I}{}^*F^\gamma \phi^{,\delta} \,,
\end{aligned} \tag{60}
$$

and ${f_2}^{\gamma\delta}$ is

$$
\begin{aligned}
{f_2}^{\gamma\delta} = & F^\gamma \phi^\delta + F^\mu \phi_{,\mu}(\phi^{,\gamma}\phi^{,\delta} - \phi_{,\mu}\phi^{,\mu}\eta^{\gamma\delta}) \\
& \left[F^\mu \phi_{,\mu}\eta^{\gamma\delta} - F^\gamma \phi^\delta + F^\mu \phi_{,\mu}{}^*F^{\gamma *}F^\delta \right] .
\end{aligned} \tag{61}
$$

Here $F^\mu = (E, 0, B)$ and ${}^*F^\mu = (B, 0, -E)$, this last as defined above.

One can in principle find solutions to equation (57) under simplified assumptions, such as considering one field a constant while the other is propagating. We will do this for the four-dimensional case in section 5.4.

5. Born-Infeld type Lagrangians in 4 dimensions

Now we consider the following theories in four dimensions: Maxwell's theory, Kaluza-Klein theory with Gauss-Bonnet term, the usual Born-Infeld electrodynamics, and the Born-Infeld theory in Kaluza-Klein space.

5.1 Maxwell's theory

The results for Maxwell's theory are well known. We choose a reduced system in order to provide a short demonstration. It is enough to consider the characteristic equations for a system depending only on two variables (t, z) and with nonzero fields given by $A_x(t,z)$, $A_y(t,z)$, $E_x(t,z)$, $E_y(t,z)$, $B_x(t,z)$ and $B_y(t,z)$, with $E_x = -\partial_0 A_x$, $E_y = -\partial_0 A_y$, $B_x = -\partial_z A_y$ and $B_y = \partial_z A_x$. We obtain the following set of four equations,

$$
\begin{aligned}
\partial_0 B_y + \partial_z E_x &= 0 \,, \\
-\partial_0 B_x + \partial_z E_y &= 0 \,, \\
\partial_z B_x - \partial_0 E_y &= 0 \,, \\
\partial_z B_y + \partial_0 E_x &= 0 \,.
\end{aligned} \tag{62}
$$

Then using the column-vector $\mathbf{u} = (E_x, E_y, B_x, B_y)$ one finds that the characteristic equation is $\det(\mathcal{A}^\mu \Sigma_\mu) = ({\Sigma_0}^2 - {\Sigma_z}^2)^2 = 0$, or in covariant form, $(g^{\mu\nu}\Sigma_\mu \Sigma_\nu)^2 = 0$. Thus there is no birefringence in Maxwell's theory, a well known fact.

5.2 The Gauss-Bonnet Lagrangian in Kaluza-Klein theory

As mentioned in the Introduction one may consider another non-linear generalization of electrodynamics that can be derived from the Kaluza-Klein theory in five dimensions, and is based on the addition of the Gauss-Bonnet term, $R_{ABCD}\, R^{ABCD} - 4\, R_{AB}\, R^{AB} + R^2$, which in five dimensions is no more a topological invariant, but leads to non-trivial equations of motion of second order when added to the usual Einstein-Hilbert Lagrangian R.

In a flat space-time and without the scalar field the Kaluza-Klein metric is

$$g_{AB} = \begin{pmatrix} g_{\mu\nu} + A_\mu A_\nu & A_\mu \\ A_\nu & 1 \end{pmatrix}, \tag{63}$$

where $A, B = 0, 1, 2, 3, 5$ and $\mu, \nu = 0, 1, 2, 3$ (or $\mu, \nu = 0, x, y, z$ following the covention we have been using). As mentioned the full Lagrangian is taken to be (see [9, 10]):

$$\mathcal{L} = R + \gamma\, (R_{ABCD}\, R^{ABCD} - 4\, R_{AB}\, R^{AB} + R^2)\,, \tag{64}$$

with γ being a certain dimensional parameter characterizing the strength of the non-linearity. When expressed in four dimensions in terms of the Maxwell tensor, it becomes

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}\, F^{\mu\nu} - \frac{3\gamma}{16}\, (F_{\mu\nu} F^{\mu\nu})^2 - 2\, (F_{\mu\lambda} F_{\nu\rho} F^{\mu\nu} F^{\lambda\rho})\,. \tag{65}$$

In terms of the invariants P and S this Lagrangian is given by $\mathcal{L} = 2P + \frac{3\gamma}{2} S^2$, which for the choice $\gamma = -\frac{2}{3}$ yields essentially the square of the Born-Infeld Lagrangian. The equations of motion are:

$$F_{\lambda\rho,\mu} + F_{\rho\mu,\lambda} + F_{\mu\lambda,\rho} = 0\,, \tag{66}$$

which correspond to the Bianchi identities and are geometrical equations valid independently of the Lagrangian chosen, and the dynamical equations resulting from the variational principle,

$$[F^{\lambda\rho} - \frac{3\gamma}{2}\, (F_{\mu\nu} F^{\mu\nu}) F^{\lambda\rho} + \frac{3\gamma}{2} F_{\mu\nu} F^{\lambda\mu} F^{\rho\nu}]_{,\lambda} = 0\,. \tag{67}$$

The Lagrangian (65) is particularly simple when expressed in more familiar terms with the fields $\mathbf{E}$ and $\mathbf{B}$:

$$\mathcal{L} = \frac{1}{2}(\mathbf{B}^2 - \mathbf{E}^2) + \frac{3\gamma}{2}\, (\mathbf{E} \cdot \mathbf{B})^2 \tag{68}$$

The equations of motion also display a clear physical meaning when expressed in terms of $\mathbf{E}$ and $\mathbf{B}$. The equation (67) becomes

$$\mathbf{div}\, \mathbf{B} = 0, \qquad \mathbf{rot}\, \mathbf{E} = -\partial_0 \mathbf{B}, \tag{69}$$

whereas the equations (68) become

$$\begin{aligned}\mathbf{div}\,\mathbf{E} &= -3\gamma\,\mathbf{B}\cdot\mathbf{grad}\,(\mathbf{E}\cdot\mathbf{B})\\ \mathbf{rot}\,\mathbf{B} &= \partial_0\mathbf{E} + 3\,\gamma\Big[\mathbf{B}\partial_0(\mathbf{E}\cdot\mathbf{B}) - \mathbf{E}\times\mathbf{grad}(\mathbf{E}\cdot\mathbf{B})\Big]\,.\end{aligned} \tag{70}$$

which show how the density of charge and the current are created by the non-linearity of the field: indeed, we can introduce

$$\rho = -3\gamma\,\mathbf{B}\cdot\mathbf{grad}\,(\mathbf{E}\cdot\mathbf{B}) \qquad \text{and} \qquad \mathbf{j} = 3\,\gamma\Big[\mathbf{B}\partial_0(\mathbf{E}\cdot\mathbf{B}) - \mathbf{E}\times\mathbf{grad}(\mathbf{E}\cdot\mathbf{B})\Big] \tag{71}$$

which satisfy the continuity equation

$$\partial_0\rho + \mathbf{div}\,\mathbf{j} = 0 \tag{72}$$

The Poynting vector conserves its form known from the Maxwellian theory, but the energy density is modified:

$$\mathbf{S} = \mathbf{E}\times\mathbf{B}, \qquad \mathcal{E} = \frac{1}{2}\,(\mathbf{E}^2 + \mathbf{B}^2) + \frac{3\gamma}{2}\,(\mathbf{E}\cdot\mathbf{B})^2\,, \tag{73}$$

with the continuity equation resuming the energy conservation satisfied by virtue of the equations of motion:

$$\partial_0\mathcal{E} + \mathbf{div}\mathbf{S} = 0\,. \tag{74}$$

The properties of possible stationary axisymmetric solutions, endowed with non-vanishing charge, intrinsic kinetic and magnetic moments, have been discussed in [9, 10]. In the theory based on the Gauss-Bonnet term in 5 dimensions, whose main features were developed in section 1, the first two equations of (62) are the same here, but the dynamical equations are instead given by

$$\begin{aligned}&(1-{E_x}^2)\partial_z B_x - (1+{B_y}^2)\partial_0 E_y - B_y B_x\partial_0 E_x - E_x B_y\partial_0 B_x - E_y B_y\partial_0 B_y\\ &\qquad -E_x B_x\partial_z E_x - E_x E_y\partial_z B_y - E_x B_y\partial_z E_y = 0\,,\\ &(1-{E_y}^2)\partial_z B_y + (1+{B_x}^2)\partial_0 E_x + E_x B_x\partial_0 B_x + B_x B_y\partial_0 E_y + E_y B_x\partial_0 B_y\\ &\qquad -E_y B_x\partial_z E_x - E_x E_y\partial_z B_y - E_y B_y\partial_z E_y = 0\,.\end{aligned} \tag{75}$$

Using again the column-vector $\mathbf{u} = (E_x, E_y, B_x, B_y)$ one finds that the characteristic equation is now

$$\begin{aligned}&\det(\mathcal{A}^\mu\Sigma_\mu) = \left({\Sigma_0}^2 - {\Sigma_z}^2\right)\times\\ &\Big[({\Sigma_0}^2 - {\Sigma_z}^2) + ({B_x}^2 + {B_y}^2){\Sigma_0}^2 + 2(E_x B_y - E_y B_x)\Sigma_0\Sigma_z + ({E_x}^2 + {E_y}^2){\Sigma_z}^2\Big]\\ &= 0\,.\end{aligned} \tag{76}$$

This can be written in covariant form in a compact manner

$$(g^{\mu\nu}\Sigma_\mu\Sigma_\nu)\,[(g^{\mu\nu} - {}^*{F^\mu}_\alpha{}^*F^{\mu\alpha})\Sigma_\mu\Sigma_\nu] = 0\,, \tag{77}$$

where ${}^*F^{\alpha\beta} \equiv \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}$ and $\epsilon^{\alpha\beta\gamma\delta}$ is the Levi-Civita tensor. Thus from (64) we can see that there is birefringence. One wave propagates in a Maxwellian way, the other propagates differently, in fact, it is delayed.

Indeed, if we set that the wavefront has the form $\Sigma = ae^{i(\omega t - kz)}$, we find two solutions given by

$$k = \pm\omega\,, \tag{78}$$

and

$$k = \pm\omega\frac{\sqrt{(E\times B)^2 + (1+B^2)(1-E^2)} \mp (E\times B)}{1-E^2}\,. \tag{79}$$

For (79) one has that the wave velocity ω/k is in general less than one, and it lags behind the other wave.

5.3 The Born-Infeld theory

Now, in four dimensions there is a new invariant, S entering the Born-Infeld action, which is

$$\mathcal{L} = \sqrt{1+2P-S^2}\,, \tag{80}$$

where, $P = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ and $S = \frac{1}{4}{}^*F_{\mu\nu}F^{\mu\nu}$, with ${}^*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ being the dual of $F_{\mu\nu}$. In a Lorentz frame one has that the action (68) is given by

$$\mathcal{L} = \sqrt{1+(\mathbf{B}^2-\mathbf{E}^2)-(\mathbf{E}\cdot\mathbf{B})^2}. \tag{81}$$

For the restricted set of fields the local equations ${}^*F^{\alpha\beta}{}_{,\alpha}$ yield the same two first equations as in (62). The dynamical equation arise from varying the action (68) with respect to A_α giving

$$\mathcal{L}^2F^{\alpha\beta}{}_{,\alpha} - \lambda_\alpha F^{\alpha\beta} + \rho_\alpha{}^*F^{\alpha\beta} = 0\,, \tag{82}$$

where $\lambda_\alpha \equiv P_{,\alpha} - SS_{,\alpha}$ and $\rho_\alpha \equiv S\lambda_\alpha - L^2S_{,\alpha}$. If we develop (70) for $\beta = x, y$ we obtain the following equations,

$$\begin{aligned}
&\left[1+{B_y}^2-{E_y}^2-{B_x}^2(B^2-E^2)-(E\cdot B)^2\right]\partial_0E_x\\
&+\left[1+{B_x}^2-{E_x}^2+{E_y}^2(B^2-E^2)-(E\cdot B)^2\right]\partial_zB_y\\
&+\left[(E_xB_y+E_yB_x)+(E\cdot B)(E_xE_y+B_xB_y)+(B^2-E^2)E_yB_x\right]\partial_zE_x\\
&+\left[-(E_xB_y+E_yB_x)+(E\cdot B)(E_xE_y+B_xB_y)-(B^2-E^2)E_yB_x\right]\partial_0B_y\\
&+\left[-2E_xB_x+(E\cdot B)({E_x}^2+{B_x}^2)-(B^2-E^2)E_xB_x\right]\partial_0B_x\\
&+\left[(E_xE_y-B_xB_y)+(E\cdot B)(E_xB_y-E_yB_x)+(B^2-E^2)E_xE_y\right]\partial_zB_x\\
&+\left[(E_xE_y-B_xB_y)+(E\cdot B)(E_xB_y-E_yB_x)-(B^2-E^2)B_xB_y\right]\partial_0E_y\\
&+\left[2E_yB_y+(E\cdot B)({E_y}^2+{B_y}^2)+(B^2-E^2)E_yB_y\right]\partial_zE_y = 0\,,
\end{aligned} \tag{83}$$

and,

$$\left[1+{B_x}^2-{E_x}^2-{B_y}^2(B^2-E^2)-(E\cdot B)^2\right]\partial_0E_y$$

$$
\begin{aligned}
&-\left[1+{B_y}^2-{E_y}^2+{E_x}^2(B^2-E^2)-(E\cdot B)^2\right]\partial_z B_x \\
&+\left[-(E_xB_y+E_yB_x)+(E\cdot B)(E_xE_y+B_xB_y)-(B^2-E^2)E_xB_y\right]\partial_0 B_x \\
&+\left[-(E_xE_y-B_xB_y)+(E\cdot B)(-E_xB_y+E_yB_x)-(B^2-E^2)B_xB_y\right]\partial_0 E_x \\
&+\left[-2E_yB_y+(E\cdot B)({E_y}^2+{B_y}^2)-(B^2-E^2)E_yB_y\right]\partial_0 B_y \\
&+\left[-(E_xE_y-B_xB_y)+(E\cdot B)(E_xB_y-E_yB_x)-(B^2-E^2)E_xE_y\right]\partial_z B_y \\
&-\left[(E_xB_y+E_yB_x)+(E\cdot B)(E_xE_y+B_xB_y)+(B^2-E^2)E_xB_y\right]\partial_z E_y \\
&-\left[2E_xB_x+(E\cdot B)({E_x}^2+{B_x}^2)+(B^2-E^2)E_xB_x\right]\partial_z E_x=0\,. \qquad (84)
\end{aligned}
$$

Then using the column-vector $\mathbf{u}=(E_x,E_y,B_x,B_y)$ one finds that the characteristic equation $\det(\mathrm{A}^\mu\Sigma_\mu)=0$ is a quartic equation in Σ_0 and in Σ_z. Expanding this equation and using Boillat's results one finds that the characteristic equation is

$$
\left[(P+1)g^{\alpha\beta}+\tau^{\alpha\beta}\right]\left[(P-1)g^{\gamma\delta}+\tau^{\gamma\delta}\right]\Sigma_\alpha\Sigma_\beta\Sigma_\gamma\Sigma_\delta=0\,, \qquad (85)
$$

where $\tau^{\alpha\beta}=Pg^{\alpha\beta}-F^{\alpha\rho}{F^\beta}_\rho$ is the Maxwellian energy-momentum tensor. We see that in principle there is birefringence, i.e four solutions (two pairs of advanced and retarded waves) for the Born-Infeld theory. However, Boillat [1] shows that if the *weak energy condition* is obeyed then the only valid characteristic equation is given by

$$
\left[(P+1)g^{\alpha\beta}+\tau^{\alpha\beta}\right]\Sigma_\alpha\Sigma_\beta=0\,. \qquad (86)
$$

5.4 Born-Infeld theory in Kaluza-Klein space

In five Kaluza-Klein dimensions one has to evaluate the determinant:

$$
\det\begin{pmatrix}
1 & {F^0}_x & {F^0}_y & {F^0}_z & \partial^0\phi \\
{F^x}_0 & 1 & {F^x}_y & {F^x}_z & \partial^x\phi \\
{F^y}_0 & {F^y}_x & 1 & {F^y}_z & \partial^y\phi \\
{F^z}_0 & {F^z}_x & {F^z}_y & 1 & \partial^z\phi \\
\partial_0\phi & \partial_x\phi & \partial_y\phi & \partial_z\phi & 1
\end{pmatrix}=
$$

$$
=1+\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}-\tfrac{1}{16}{}^*F_{\mu\nu}F^{\mu\nu}-\phi_{,\mu}\phi^{,\mu}-\phi_{,\mu}\phi_{,\nu}(\tfrac{1}{2}F_{\alpha\beta}F^{\alpha\beta}g^{\mu\nu}-F^{\mu\alpha}{F^\nu}_\alpha)\,. \qquad (87)
$$

Thus

$$
\mathcal{L}=\sqrt{1+2P-S^2-2\Phi-2I}\,. \qquad (88)
$$

As before, $P=\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, $S=\frac{1}{4}{}^*F_{\mu\nu}F^{\mu\nu}$, $\Phi=\frac{1}{2}\phi_{,\mu}\phi^{,\mu}$, and $I=\frac{1}{2}\phi_{,\mu}\phi_{,\nu}(\frac{1}{2}F_{\alpha\beta}F^{\alpha\beta}g^{\mu\nu}-F^{\mu\alpha}{F^\nu}_\alpha)$.

For the full case one encounters huge and intractable equations. In order to simplify the problem we suppose a constant electromagnetic background $F_{\mu\nu}=$ a constant tensor. In this case the Lagrangian reduces to

$$
\mathcal{L}=\sqrt{1+2\vartheta}\,, \qquad (89)
$$

where $\vartheta=\frac{1}{2}\Theta^{\mu\nu}\phi_{,\mu}\phi_{,\nu}$ and $\Theta^{\mu\nu}$ is also a constant and symmetric tensor given by

$$
\Theta^{\mu\nu}=\frac{F^{\mu\alpha}\,{F^\nu}_\alpha-(1+2P)g^{\mu\nu}}{\sqrt{1+2P-S^2}} \qquad (90)
$$

Then the equation of motion for ϕ is

$$(1+2\vartheta)\Theta^{\alpha\beta}\phi_{,\beta,\alpha} - \Theta^{\mu\nu}\phi_{,\mu}\phi_{\alpha,\nu}\Theta^{\alpha\beta}\phi_{,\beta} = 0\,. \tag{91}$$

We suppose now that the scalar field ϕ propagates in the (t,z) directions with $\psi = \partial_0\phi$ and $\chi = \partial_z\phi$. We are also assuming, as before, $E_z = 0$ and $B_z = 0$. In addition, by choosing appropriately the directions of the fields $\mathbf{E}$ and $\mathbf{B}$ one can study a situation in which the relevant conponents of $\Theta^{\mu\nu}$ are Θ^{00} and Θ^{zz} alone, with $\Theta^{0z} = 0$. This can be achieved in three distinct situations: (i) $E_x = E_y = 0$, (ii) $E_x = B_x = 0$, or (iii) $E_y = B_y = 0$. Then, $\Theta^{00} = -\frac{1+\mathbf{B}^2}{\sqrt{1+2P-S^2}}$, and $\Theta^{zz} = \frac{1-\mathbf{E}^2}{\sqrt{1+2P-S^2}}$ and Θ^{xx}, Θ^{yy} are non-zero but not relevant to the problem. To simplify the notation we write $\Theta^{00} = l$ and $\Theta^{zz} = m$. With these assumptions the equation of motion takes the form

$$-l\,\partial_0\psi + m\partial_z\chi - lm\,\psi(\psi\partial_z\chi - \chi\partial_z\psi) - lm\,\chi(\chi\partial_0\psi - \psi\partial_0\chi)\,. \tag{92}$$

Taking into account the internal equation $\partial_0\chi - \partial_z\psi = 0$, the matrix $\mathcal{A}^\mu\Sigma_\mu$ is

$$\mathcal{A}^\mu\Sigma_\mu = \begin{pmatrix} -\Sigma_z & \Sigma_0 \\ -l\,(1+m\chi^2)\,\Sigma_0 + lm\,\psi\chi\,\Sigma_z & lm\,\psi\chi\,\Sigma_0 + m\,(1-l\,\psi^2)\,\Sigma_z \end{pmatrix}. \tag{93}$$

Its determinant yields the following characteristic equation

$$l\,{\Sigma_0}^2 - m\,{\Sigma_z}^2 + lm\,(\chi^2\,{\Sigma_0}^2 - 2\psi\chi\,\Sigma_0\,\Sigma_z + \psi^2\,{\Sigma_z}^2) = 0\,. \tag{94}$$

Equation (94) can be written in a covariant form

$$-\Theta^{\mu\nu}\,\Sigma_\mu\,\Sigma_\nu + (\Theta^{\alpha\mu}\,\Theta^{\beta\nu}\phi_{,\alpha}\phi_{,\beta} - 2\vartheta\,\Theta^{\mu\nu})\,\Sigma_\mu\,\Sigma_\nu = 0\,. \tag{95}$$

In vacuum $\Theta^{\mu\nu} = -g^{\mu\nu}$ and (95) reduces to the propagation of a scalar Born-Infeld field $g^{\mu\nu}\,\Sigma_\mu\Sigma_\nu + (\phi^{,\mu}\phi^{,\nu} - g^{\mu\nu}\,\phi_{,\rho}\phi^{,\rho})\,\Sigma_\mu\,\Sigma_\nu = 0$.

In order to find $k(\omega)$ we put the ansatz $\Sigma = a\mathrm{e}^{i(\omega t - kz)}$ into equation (94) yielding

$$k = \omega\sqrt{\frac{l}{m}}\,\frac{\sqrt{1 - l\,\psi^2 + m\,\chi^2} - \sqrt{lm}\,\psi\chi}{1 - l\,\psi^2}\,. \tag{96}$$

One can study various limits. In particular, an interesting case occurs when $\psi^2 \sim \chi^2 \sim \psi\chi \sim 0$ and $E^2 \sim B^2 << 1$. Then the velocity of propagation of the wave is $v = \frac{\omega}{k} = \sqrt{\frac{l}{m}} = 1 - \epsilon$ with $\epsilon << 1$. In this example we recover the propagation of a linear scalar field in smooth electromagnetic background.

6. Conclusion

In the present article we have shown how various multi-dimensional generalizations of the Born-Infeld theory lead to complicated interactions between the electromagnetic and dilaton fields. The exceptional character of this theory is lost as soon as the dilaton field appears, induced by the extra Kaluza-Klein dimension. Only in very low dimensions the birefringence is not observed; in four space-time dimensions it appears in all the generalizations of the Born-Infeld theory.

Considering the Born-Infeld theory as derived from an effective Lagrangian of a multi-dimensional string Lagrangian, we come to the conclusion that the features that were particularly interesting in classical version are lost due to the emergence of highly non-linear couplings between all the fields present in the theory.

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